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Multigrid Method for the Numerical Solution of Parabolic Partial Differential Equations using Biorthogonal Wavelets

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Abstract

In this paper, we proposed multi-grid method for the numerical solution of parabolic partial differential equations (PDEs) using biorthogonal wavelets. The standard multigrid procedure performs poorly or may break down when used to solve certain PDEs with discontinuous or highly oscillatory coefficients and also involve some difficulty to observe fast convergence in low computational time. To overcome this, we used Biorthogonal Wavelet Based Multigrid Method for solving parabolic PDEs in which the system of equations arising from the finite difference discretization is represented in wavelet bases. Some of the test problems are presented to demonstrate the validity and applicability of the proposed method.

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1. Introduction

A variety of problems in physics, chemistry and biology have their mathematical setting as linear and nonlinear partial differential equations (PDEs). Many of the phenomena that arise in mathematical physics and engineering fields can be described by PDEs. In physics for example, the heat flow and the wave propagation phenomena are well described by PDEs. Moreover, most physical phenomena of fluid dynamics, quantum mechanics, electricity, plasma physics, propagation of shallow water waves, and many other models are formulated by PDEs [14]. Due to these huge applications, there is a demand on the development of accurate and efficient analytical or numerical methods able to deal with these equations. Except for a few number of these problems, we encounter difficulties in finding their analytical solutions. Many attempts have been made to develop numerical methods to solve the linear and nonlinear PDEs, see [12, 15].

There are several applications of parabolic PDEs in science and engineering. Also many reaction–diffusion problems in biology and chemistry are modeled by parabolic PDEs. Analytical solution of certain parabolic

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PDEs either does not exist or is hard to find. Due to this fact, in the last decades, there have been great advances in the development of finite difference, finite element, spectral techniques and finite volume methods for the solution of parabolic PDEs. The parabolic PDEs of the forms [4],

$$u_t = u_{xx} + f(u) \text{ or}$$
$$u_t = u_{xx} + g(x,t)\,, 0 \le x \le 1 \;\&\; t > 0$$

subject to initial condition (IC) and boundary conditions (BCs). Where and are the functions of dependent and independent variables. Finite difference methods have been commonly used for the numerical solution of boundary value problems (BVPs). To find solutions to PDEs, for most cases, it is necessary to employ discretization methods to reduce the sets of PDEs to systems of algebraic equations and such type of equations are solved by using direct methods. Direct methods are theoretically producing the exact solution to the system in a finite number of steps. In practice, of course, the solution obtained will be contaminated by the round-off error. To minimize such round-off error, iterative methods are frequently used for solving linear systems. For large systems, these methods are efficient in terms of both computer storage and computational cost. The multi-grid approach is one of the method to overcome these drawbacks was realized after the works of A. Brandt [2] and W. Hackbusch [9]. The multigrid method is largely applicable in increasing the efficiency of iterative methods used to solve large system of algebraic equations.

Recently, many authors De Leon [10] and Bujurke et al. [3, 4] have developed wavelet multigrid methods. Also Wesseling [17] introduced the multigrid method which is very useful in increasing the efficiency of iterative methods used to solve systems of algebraic equations approximating PDEs. Bastian, Burmeistier and Horton [1] was investigated in series of experiments to solve parabolic PDEs using multigrid methods. However, when meet by certain problems, for example parabolic type of problems with discontinuous or highly oscillatory coefficients, as well as advection-dominated problems, the standard multigrid procedure converges slowly with larger computational time or may break down. For this reason, we go for wavelet multigrid method in which by choosing the filter operators obtained from wavelets to define the prolongation and restriction operators.

"Wavelets" have been very popular topic of conversations in many scientific and engineering gatherings these days. Some of the researchers have decided that, wavelets as a new basis for representing functions, as a technique for time-frequency analysis, and as a new mathematical subject. Of course, "wavelets" is a versatile tool with very rich mathematical content and great potential for applications. However, wavelet analysis is a numerical concept which allows one to represent a function in terms of a set of bases functions, called wavelets, which are localized both in location and scale. In wavelet applications to the solution of partial differential equations the most frequently used wavelets are those with compact support introduced by Daubechies [10]. Several studies explored the usage of Daubechies wavelets to solve partial differential equations [5, 8].

This paper gives an alternative method i.e. Biorthogonal wavelet based multigrid method (BWMGM) for the numerical solution of parabolic PDEs. The BWMGM formulated in this paper have the following characteristics.

- They provide approximations which are continuous and continuously differentiable throughout the domain of the problems, and have piecewise continuous second derivatives.
- The methods possess super convergence properties.
- The methods incorporate IC and BCs in a systematic fashion.

The organization of the paper is as follows. Preliminaries of wavelets are given in section 2. Section 3 describes the method of solution. Numerical findings and error analysis are presented in section 4. Finally, conclusion of the proposed work is discussed in section 5.

2. Properties of Biorthogonal Wavelets

The framework of the theory of orthonormal wavelets to the case of biorthogonal wavelets by a modification of the approximation space structure is extended by Cohen et al. [6]. In [11], Ruch and Fleet build a biorthogonal structure called dual multiresolution analysis that allows for the construction of symmetric scaling filters and that can incorporate spline functions. They used instead of scaling $\{a_n\}$ and wavelet $\{b_n\}$ filters, the new construct yields scaling $\{\tilde{a}_n\}$ and wavelet $\{\tilde{b}_n\}$ filters as decomposition and reconstruction. Instead of a single scaling function $\phi(x)$ and wavelet function $\psi(x)$, the dual multiresolution analysis requires a pair of scaling functions $\phi(x)$ and $\tilde{\phi}(x)$ related by a duality condition similarly, a pair of wavelet functions $\psi(x)$ and $\bar{\Psi}(x)$. To construct the BDWT matrix, the same thing is used in to build the orthogonal discrete wavelet transform matrix. Due to excellent properties of biorthogonality and minimum compact support, CDF wavelets can be useful and convenient, providing guaranty of convergence and accuracy of the approximation in a wide variety of situations.

Let's consider the (5, 3) biorthogonal spline wavelet filter pair, We have $\tilde{a} = (\tilde{a}_{-1}, \tilde{a}_0, \tilde{a}_1) = \left(\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

and $a = (a_{-2}, a_{-1}, a_0, a_1, a_2) = \left(\frac{-1}{4\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{-1}{4\sqrt{2}}\right)$

To form the high pass filters. We have $b_k = (-1)^k \tilde{a}_{1-k}$ and $\tilde{b}_k = (-1)^k a_{1-k}$.

The high pass filter pair b_k and $\tilde{b}_k$ for the (5, 3) biorthogonal spline filter pair.

$b_0 = \frac{1}{2\sqrt{2}},\ b_1 = \frac{-1}{\sqrt{2}},\ b_2 = \frac{1}{2\sqrt{2}}$ and $\tilde{b}_{-1} = \frac{1}{4\sqrt{2}},\ \tilde{b}_0 = \frac{1}{2\sqrt{2}},\ \tilde{b}_1 = \frac{-3}{2\sqrt{2}},\ \tilde{b}_2 = \frac{1}{2\sqrt{2}},\ \tilde{b}_3 = \frac{1}{4\sqrt{2}}$

In this paper, we use the filter coefficients which are,

Low pass filter coefficients: $a_{-2}, a_{-1}, a_0, a_1, a_2$ and High pass filter coefficients: b_0, b_1, b_2 for decomposition matrix. Low pass filter coefficients: $\tilde{a}_{-1} = b_2,\ \tilde{a}_0 = -b_1,\ \tilde{a}_1 = b_0$ and High pass filter coefficients: $\tilde{b}_{-1} = -a_2,\ \tilde{b}_0 = a_1,\ \tilde{b}_1 = -a_0,\ \tilde{b}_2 = a_{-1},\ \tilde{b}_3 = -a_{-2}$ for reconstruction matrix.

Discrete wavelet transforms (DWT): The matrix formulation of the discrete signals and DWT play an important part in the wavelet method. As we already know about the DWT matrix and its applications in the wavelet method given and are explained in [13].

3. Biorthogonal Spline Wavelet Operators

Using these matrices, we introduced biorthogonal spline wavelet restriction and biorthogonal spline wavelet prolongation operators respectively. i.e.,

$$BSW_R = \begin{pmatrix} a_{-1} & a_0 & a_1 & a_2 & 0 & 0 & 0 & \cdots & 0 & a_{-2} \\ b_0 & b_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_{-1} \\ 0 & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & b_{-1} & b_0 & b_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & \ddots & & \cdots & & \cdots & & 0 & 0 \\ 0 & 0 & \cdots & 0 & b_{-1} & b_0 & b_1 & 0 & \cdots & 0 & 0 \end{pmatrix}_{\frac{N}{2}\times N}$$

and

$$BSW_P = \begin{pmatrix} \tilde{a}_0 & \tilde{a}_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \tilde{a}_{-1} \\ \tilde{b}_0 & \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 & 0 & 0 & 0 & \cdots & 0 & \tilde{b}_{-1} \\ 0 & \tilde{a}_{-1} & \tilde{a}_0 & \tilde{a}_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \tilde{b}_{-1} & \tilde{b}_0 & \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \ddots & & \ddots & & & \vdots \\ 0 & \cdots & 0 & 0 & \tilde{b}_{-1} & \tilde{b}_0 & \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 & 0 & \cdots & 0 \end{pmatrix}_{N\times\frac{N}{2}}$$

4. Method of Solution

In this section, we applied the wavelet multigrid method for the numerical solution of linear parabolic PDEs. as follows:

4.1. Multigrid (MG) Method

A linear partial differential equation of parabolic type is of the form

$$\left.\begin{aligned} u_t &= u_{xx} + f(u) \quad or \\ u_t &= u_{xx} + g(x,t),\ \ 0 \le x \le 1\ \ \&\ t > 0 \end{aligned}\right\} \quad (4.1)$$

subject to initial condition (IC) and boundary conditions (BCs). Where $g(x,t)$ is the function of independent variables.

Now consider the Eq. (4.1),using finite difference scheme, discretizing the PDEs into a system of algebraic equations. This can be written as follows

$$\mathrm{Au} = \mathrm{b} \quad (4.2)$$

Where, A is $N \times N$ coefficient matrix, b is $N \times 1$ matrix and u is $N \times 1$ matrix to be determined. Solving Eq. (4.2) through the iterative method, we get the approximate solution v of u. i.e., $\mathrm{u} = \mathrm{e} + \mathrm{v} \Rightarrow \mathrm{v} = \mathrm{u} - \mathrm{e}$, where e is ($N \times 1$ matrix) error to be determined.

In the computation of numerical analysis, approximate solution containing some error. There are many approaches to minimize the error. Some of them are Multigrid (MG) and Wavelet-multigrid (WMG) methods. Now, we are discussing the multigrid method of solution as follows,

From Eq. (4.2), we get the approximate solution v for u. Now we first find the residual as

$$\mathrm{r}_{N\times 1} = [\mathrm{b}]_{N\times 1} - [\mathrm{A}]_{N\times N}[\mathrm{v}]_{N\times 1} \quad (4.3)$$

We reduce the matrices in the finer level to coarsest level using Restriction operator and then construct the matrices back to finer level from the coarsest level using Prolongation operator.
Reduce the matrices in the finer level to coarsest level using Restriction operator as

$$R_O = \frac{1}{4}\begin{pmatrix} 1 & 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & & \dots & 0 & 0 \\ 0 & 0 & & \dots & & \dots & 1 & 2 \end{pmatrix}_{N/2\times N}$$

and then construct the matrices back to finer level from the coarsest level using Prolongation operator as.

$$P_O = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 2 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & \vdots \\ 0 & 2 & \vdots & \dots & \vdots \\ 0 & 1 & & & \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix}_{N\times N/2}$$

From Eq. (4.3), $r_{N/2\times1} = [R_O]_{N/2\times N}[r]_{N\times1}$ (4.4)

and $[A]_{\frac{N}{2}\times\frac{N}{2}} = [R_O]_{\frac{N}{2}\times N}[A]_{N\times N}[P_O]_{N\times\frac{N}{2}}$

solving, $[\mathrm{A}]_{\frac{N}{2}\times\frac{N}{2}}[\mathrm{e}]_{\frac{N}{2}\times1} = [\mathrm{r}]_{\frac{N}{2}\times1}$

we get the residual equation, $[\mathrm{e}]_{\frac{N}{2}\times1}$ with initial guess '0'.

From Eq. (4.4), $\mathrm{r}_{\frac{N}{4}\times1} = [R_O]_{\frac{N}{4}\times\frac{N}{2}}[\mathrm{r}]_{\frac{N}{2}\times1}$ (4.5)

and $[\mathrm{A}]_{\frac{N}{4}\times\frac{N}{4}} = [R_O]_{\frac{N}{4}\times\frac{N}{2}}[\mathrm{A}]_{\frac{N}{2}\times\frac{N}{2}}[P_O]_{\frac{N}{2}\times\frac{N}{4}}$

solving, $[\mathrm{A}]_{\frac{N}{4}\times\frac{N}{4}}[\mathrm{e}]_{\frac{N}{4}\times1} = [\mathrm{r}]_{\frac{N}{4}\times1}$

again, we get the residual equation, $[\mathrm{e}]_{\frac{N}{4}\times1}$ with initial guess '0'.

So on, the procedure is continuing up to the coarsest level, we have,

$$\mathrm{r}_{1\times1} = [R_O]_{1\times2}[\mathrm{r}]_{2\times1} \quad (4.6)$$

And $[\mathrm{A}]_{1\times1} = [R_O]_{1\times2}[\mathrm{A}]_{2\times2}[P_O]_{2\times1}$

solving, $[\mathrm{A}]_{1\times1}[\mathrm{e}]_{1\times1} = [\mathrm{r}]_{1\times1}$

we get the residual equation, $[\mathrm{e}]_{1\times1}$ exactly.

Now correct the solution, i.e., $\mathrm{u}_{2\times2} = [\mathrm{e}]_{2\times1} + [P_O]_{2\times1}[\mathrm{e}]_{1\times1}$

solving, $[\mathrm{A}]_{2\times2}[\mathrm{u}]_{2\times1} = [\mathrm{r}]_{2\times1}$, we get $\mathrm{u}_{2\times1}$.

again, correct the solution $u_{4\times1} = [e]_{4\times1} + [P_0]_{4\times2}[u]_{2\times1}$

solving, $[\mathrm{A}]_{4\times4}[\mathrm{u}]_{4\times1} = [\mathrm{r}]_{4\times1}$, we get $\mathrm{u}_{4\times1}$.

So on, continuing the procedure up to the finer level,

Lastly, correct the solution, $u_{N\times1} = [v]_{N\times1} + [P_0]_{N\times\frac{N}{2}}[P_0]_{\frac{N}{2}\times1}$

solving, $[\mathrm{A}]_{N\times N}[\mathrm{u}]_{N\times1} = [\mathrm{b}]_{N\times1}$, we get $\mathrm{u}_{N\times1}$.

where, $\mathrm{u}_{N\times 1}$ is the required approximate solution of the system Eq. (4.2).

4.2. Wavelet Multigrid Method (WMGM)

The same procedure is applied as explained the MG method (Section 4.1) in which replacing operators BSW_R and BSW_P in place of R_O and P_O respectively.

5. Numerical Experiments

In this section, we applied multigrid for the numerical solution of parabolic partial differential equations using Biorthogonal wavelets and subsequently presented the efficiency of the methods in the form of tables and figures. The error analysis is considered as $E_{max} = \max|u_e - u_a|$, where u_e and u_a are exact and approximate solutions respectively.

Test problem 5.1. Consider homogeneous parabolic PDE [2],

$$u_t = u_{xx} - 2u \quad (5.1)$$

with initial condition : $u(x,0) = \sinh x\ ,\ 0 \le x < 1$ (5.2)

and boundary conditions: $u(0,\ t) = 0,\quad u(1,\ t) = \sinh(1)\times e^{-t} \quad t > 0$ (5.3) Which has the exact solution $u(x,t) = \sinh x \times e^{-t}$ .By applying the methods explained in the section 4, we obtain the numerical solutions and compared with exact solution are presented in figure 1. The maximum absolute errors with CPU time of the methods are presented in table 1.

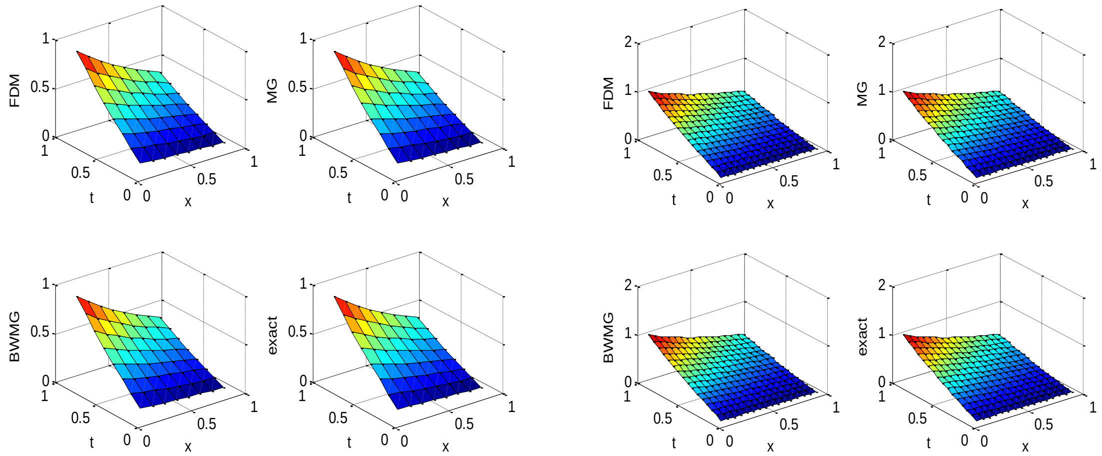


Fig. 1. Comparison of numerical solutions with exact solution of test problem 5.1 for N=64 & 256.

Table 1. Maximum error and CPU time (in seconds) of the methods of test problem 5.1.

N	Method	$E_{\max}$	Setup time	Running time	Total time
16	FDM	4.1408e-03	0.0771	4.4553	4.5324
	MG	4.1408e-03	0.0035	0.1411	0.1446
	BWMG	4.1408e-03	0.0204	0.0415	0.0619
64	FDM	2.3758e-03	0.0902	3.0709	3.1611
	MG	2.3758e-03	0.0045	0.1581	0.1626
	BWMG	2.3758e-03	0.0025	0.0525	0.0550
256	FDM	1.2959e-03	0.1532	4.0626	4.2158
	MG	1.2959e-03	0.0045	0.2931	0.2976
	BWMG	1.2959e-03	0.0022	0.1883	0.1905
1024	FDM	6.8050e-04	1.1261	6.6798	7.8059
	MG	6.8050e-04	0.0285	2.4989	2.5274
	BWMG	6.8050e-04	0.0117	0.6257	0.6374

Test problem 5.2. Next**, c**onsider the non-homogeneous parabolic PDE [2],

$$u_t = u_{xx} + \cos x \qquad (5.2)$$

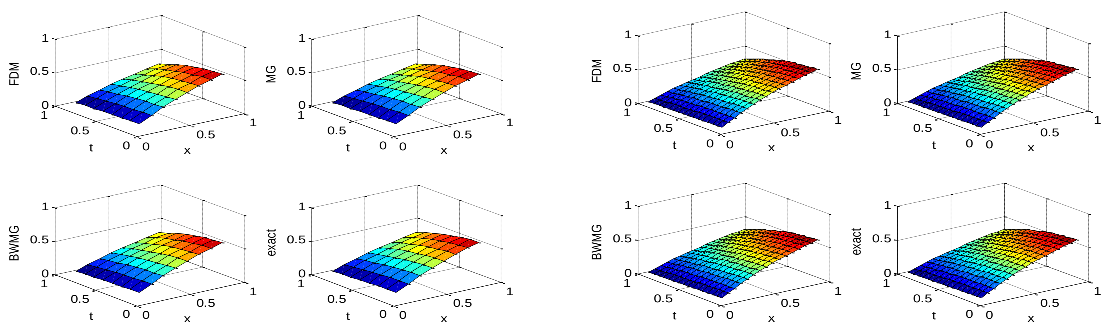
with initial condition:

$$u(x,0) = 0\ ,\ 0 \le x < 1 \qquad (5.3)$$

and boundary conditions:

$$u(0,\,t) = 1 - e^{-t}, \quad u(1,\,t) = \cos(1) \times \left(1 - e^{-t}\right) \quad t > 0 \qquad (5.4)$$

which has the exact solution $u(x,t) = \cos x \times (1 - e^{-t})$. By applying the methods explained in the section 4, we obtain the numerical solutions and compared with exact solution are presented in figure 2. The maximum absolute errors with CPU time of the methods are presented in table 2.



N = 64 ***N = 256***

Fig. 2. Comparison of numerical solutions with exact solution of test problem 5.2 for N=64 & 256.

Table 2. Maximum error and CPU time (in seconds) of the methods of test problem 5.2.

N	Method	$E_{\max}$	Setup time	Running time	Total time
16	FDM	6.9470e-03	0.0771	6.9939	7.071
	MG	6.9470e-03	0.0034	0.1433	0.1467
	BWMG	6.9470e-03	0.0020	0.0419	0.0439
64	FDM	4.1792e-03	0.0892	3.6699	3.7591
	MG	4.1792e-03	0.0055	0.1932	0.1987
	BWMG	4.1792e-03	0.0025	0.0519	0.0544
256	FDM	2.3120e-03	0.1417	3.5273	3.669
	MG	2.3120e-03	0.0046	0.3194	0.324
	BWMG	2.3120e-03	0.0034	0.1900	0.1934
1024	FDM	1.2212e-03	0.9679	5.7661	6.734
	MG	1.2212e-03	0.0061	2.4902	2.4963
	BWMG	1.2212e-03	0.0032	0.6757	0.6789

5. Conclusions

In this paper,BWMG (biorthogonal wavelet based multigrid) method for the numerical solution of parabolic PDEs using spline filter coefficients has been presented. From the above figures and tables, BMGM shows significant advantages over the existing methods i.e. FDM and MGM. i.e. the test problems, shows the numerical solutions obtained agrees with the exact solution, but CPU time of the proposed scheme is lesser than the existing methods. Also the convergence of the presented methods is observed i.e. the error decreases when the level of resolution Nincreases. It is worth mentioning that this method is capable of reducing the volume of the computational work as compared to the classical methods and is still maintaining the high accuracy of numerical result. Hence the proposed scheme BWMGM is very effective for solving linear partial differential equations.

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